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Kinks in General Relativity

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The problem of classifying topologically distinct general relativistic metrics is discussed. For a wide class of parallelizable space-time manifolds it is shown that a certain integer-valued topological invariant n always exists, and that quantization when n is odd will lead to spinor wave functionals.

1. INTRODUCTION

The possibility of field theories allowing configurations that are topologically distinct and that cannot be continuously deformed into the vacuum field is an idea that has found application in a number of areas of physics. A field configuration represents a mapping from the domain X of the field variables into the range Y, where we shall assume Y to be a connected manifold. There is usually a restriction that the only mappings considered are those that map some fixed base point (or points) $x_0 \in X$ into a fixed base point $y_0 \in Y$. In studying such field theories it is thus important to be able to list the topologically distinct classes of base-point-

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preserving mappings $\varphi: X \to Y$. The set of all such classes, called homotopy classes, is denoted by $[X, Y]$. The determination of $[X, Y]$ for a given X and Y is the well-known classification problem of homotopy theory.

For many field theories of physical interest X is three-dimensional Euclidean space R^3, with the infinite boundary of R^3 mapping into y_0. Under these circumstances, R^3 can be replaced by its one-point compactification, the 3-sphere, S^3. In studying different field theories of this type one is therefore interested in calculating $[X, Y]$ for different examples of Y, with $X = S^3$.

One of the earliest field theories to be studied from the point of view of homotopy theory was general relativity. Here the classification problem is turned around. Y, as we shall explain, is fixed and one is interested in computing $[X, Y]$ for different examples of the space-time manifold X. Finkelstein and Misner (1959) analyzed the case in which the space-time manifold is chosen to be $R^3 \times R^1$ (or equivalently $S^3 \times R^1$). They found that the homotopy classes of metric could be specified by a single integer label, n. For this special case Williams and Zvengrowski (1977) have shown that the $n = 1$ configurations correspond to fermions and that spinor wave functionals can be defined.

The present paper discusses the classification problem for a general parallelizable space-time manifold and the mechanism by which half-odd-integer spin can arise. For any space-time manifold that is a bundle space whose fiber is R^1 and whose base is any closed (i.e., compact and boundaryless), connected, orientable 3-manifold it is shown in Section 4 that the homotopy classes of Lorentz metrics can be put in one-to-one correspondence with a group that is a direct sum of the group of integers Z and a (possibly zero) number of groups of integers modulo 2, Z_2; i.e. metrics are topologically classified according to $Z \oplus Z_2 \cdots \oplus Z_2$. For such space-time manifolds it is shown in Section 5 that spinor wave functionals can be defined on suitable classes of metrics labeled by odd $n \in Z$. Sections 4 and 5 are arranged so that the main results are stated at the beginning. These are then followed by proofs of a more technical nature. Some interesting examples of space-time manifolds are discussed in Section 6.

Before treating the general relativistic case, it will be advantageous to review the simpler situation of field theories whose domain is S^3. This is done in the following section.

2. KINKS WITH DOMAIN S^3

Consider a field theory described in terms of mappings

$$\varphi: R^3 \to Y$$

with

$$\varphi(\mathbf{x}) \to y_0 \qquad \text{as } |\mathbf{x}| \to \infty$$

The boundary conditions at infinity allow R^3 to be replaced by S^3, so that the classification problem is solved by computing $[S^3, Y]$, which is the third homotopy group $\pi_3(Y)$. This is Abelian for all Y. The group identity, denoted by 0 (or by Q_0), is the class of mappings containing the constant map φ_0 which maps the whole of R^3 into y_0. If the group identity is not the only element of $\pi_3(Y)$, then the theory is said to *admit kinks* (Finkelstein, 1966).

If Y is a vector space, then $\pi_3(Y)=0$ and there are no kinks. For a less trivial example, consider Skyrme's theory of strong interactions (Skyrme, 1971; Williams, 1970; Pak and Tze, 1979) in which $Y=S^3$. The theory admits kinks since $\pi_3(S^3)\approx Z$. The homotopy classes of mappings can be labeled by a single integer and we may denote them by $\ldots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \ldots$. If n is positive, mappings belonging to Q_n are called n-kink mappings and mappings belonging to Q_{-n} are called n-antikink mappings. The elements Q_1 and Q_{-1} are generators of $\pi_3(S^3)$ and the group operation works according to $Q_n + Q_m = Q_{n+m}$. A simple example of a 1-kink mapping $\varphi_1: R^3 \to S^3$ is provided by $\varphi_1(\mathbf{x}) = (\phi_1, \phi_2, \phi_3, \phi_4)$ with

$$\phi_i = 2ax_i/(r^2+a^2), \qquad i=1,2,3$$

$$\phi_4 = (r^2-a^2)/(r^2+a^2)$$

where $r=|\mathbf{x}|$ and $a \neq 0$ is a constant. This is the usual stereographic projection. The homotopy class of a mapping is denoted by square brackets. Thus $[\varphi_1] = Q_1$.

The 1-kink mappings of Skyrme's theory are degree-1 mappings. It is possible to define the degree of mapping between any two orientable manifolds of the same dimension. If $\varphi: X^n \to Y^n$, then $\deg\varphi$ is, roughly speaking, the number of times that φ wraps X^n around Y^n. More precisely, because the integral singular nth homology groups of any orientable n-manifolds satisfy $H_n(X^n) \approx H_n(Y^n) \approx Z$, any mapping $\varphi: X^n \to Y^n$ induces a homomorphism (Spanier, 1966, p. 207),

$$\varphi_*: H_n(X^n) \to H_n(Y^n)$$

The homomorphism corresponds to multiplication by some integer, and $\deg\varphi$ is defined to equal this integer (up to a sign).

Consider a theory for which $Y=P^3$, real projective 3-space (i.e., S^3 with antipodal points identified). Since $\pi_3(P^3)\approx Z$, there are kinks. Let $\kappa: S^3 \to P^3$ be the usual double covering map which identifies antipodal points of S^3. Then $[\kappa]$ generates $\pi_3(P^3)$ and so κ is a 1-kink map (or a 1-antikink map, depending on the convention chosen). However, $\deg \kappa = 2$. There are no mappings $\varphi: S^3 \to P^3$ of odd degree. The question of kink number is thus more closely related to homotopy (i.e., generating $[X, Y]$) than to homology (i.e., degree of mapping). It should also be pointed out that the above examples are simple in the sense that they admit only one kind of kink. A more complicated theory could be obtained by choosing $Y = SO(4)$, for example. Since $SO(4) = S^3 \times P^3$, it follows that $\pi_3(SO(4)) \approx Z \oplus Z$ so that there are two different types of kink.

An important feature of many kink theories is the possibility that the kinks are (classical analogs of) fermions. This happens when the double connectedness of the three-dimensional rotation group $SO(3)$ implies a corresponding double connectedness in a particular homotopy class Q_w. (We use w to denote any system of labels, not just a single integer). It is then possible to define wave functionals $\Psi: Q_w \to \mathbb{C}$ which are double valued under the action of the rotation group. ($\mathbb{C}$ denotes the set of complex numbers.) We shall continue to talk of "double-valued functionals" although strictly speaking we should be considering single-valued functionals defined on the universal covering space of Q_w.

Consider $SO(3)$. Since $\pi_1(SO(3)) \approx Z_2$, there will be two different types of loop in $SO(3)$. The generator of $\pi_1(SO(3))$ contains the loops that are nontrivial in the sense that they are not deformable to a point. The 2π rotation loops are of this type. Let I^1 denote the unit interval $[0, 1]$. A simple example of a 2π rotation loop in $SO(3)$ is given by $\lambda: I^1 \to SO(3)$ with $\lambda(t) = R_t$, where

$$R_t = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad 0 \leqslant t \leqslant 1$$

Since all 2π rotation loops in $SO(3)$ are homotopic to each other, it will be sufficient to consider this one example. Let $\varphi: R^3 \to Y$ be a mapping belonging to Q_w. Then R_t can be used to define a 2π rotation loop ρ in Q_w according to $\rho: I^1 \to Q_w$ with $\rho(t)(\mathbf{x}) = \varphi(R_t \mathbf{x})$. Either all 2π rotation loops in Q_w will be nontrivial or none of them will. If 2π rotation loops in Q_w are nontrivial, then it will be possible to define wave functionals $\Psi(\varphi)$ that are double valued and that change sign under 2π rotations. After Finkelstein and Rubinstein (1968) we make the following definition.

Definition. If, for a particular theory, the 2π rotation loops in a homotopy class Q_w are nontrivial, then Q_w (and the theory) is said to *admit half-odd-integer spin.*

Skyrme's theory has been shown to admit half-odd-integer spin (Williams and Zvengrowski, 1977). Theories with $Y=SU(3)$ do not admit half-odd-integer spin (Dowker, 1972).

3. GENERAL RELATIVISTIC KINKS

Let $\mathfrak{M}^4$ denote the four-dimensional space-time manifold of general relativity. As previously mentioned we consider only parallelizable $\mathfrak{M}^4$ in which case the tangent bundle $T(\mathfrak{M}^4)$, its dual $T^*(\mathfrak{M}^4)$, and any bundle associated with these is trivial. In particular the bundle of Lorentz metric tensors $T_{2L}(\mathfrak{M}^4)$ is trivial, where $T_{2L}(\mathfrak{M}^4)$ is defined as follows (see also Steenrod, 1951, §40.1). Take $S_{4,1}$ to be the space of all real symmetric nonsingular 4×4 matrices of signature 2, that is all matrices congruent to diag $(1,1,1,-1)$. The group $GL(4,R)$ acts on $S_{4,1}$ by congruence $\tau\to\sigma^T\tau\sigma$, where $\tau\in S_{4,1}$ and $\sigma\in GL(4,R)$. Then $T_{2L}(\mathfrak{M}^4)$ is the bundle associated to $T^*(\mathfrak{M}^4)$ with the given action of $GL(4,R)$ and with fiber $S_{4,1}$. Locally, at each point $p\in\mathfrak{M}^4$, by choosing a basis $\{\mathbf{f}_\mu\}$ for the tangent space T_p at p and letting $\{\boldsymbol{\theta}^\mu\}$ be the corresponding dual basis for T_p^*, any element $\mathbf{g}\in T_{2L}(p)$, the fiber over p, has a representation

$$\mathbf{g}=\sum g_{\mu\nu}\boldsymbol{\theta}^\mu\otimes\boldsymbol{\theta}^\nu$$

where $\|g_{\mu\nu}\|\in S_{4,1}$. We denote the projection map of the bundle by $\Pi\colon T_{2L}(\mathfrak{M}^4)\to\mathfrak{M}^4$.

By a cross-section one means a map $C\colon\mathfrak{M}^4\to T_{2L}(\mathfrak{M}^4)$ such that $\Pi{:}C=I_{\mathfrak{M}^4}$, the identity map of $\mathfrak{M}^4$. Since the bundle is trivial a cross section always exists. One calls a cross section C a Lorentz metric tensor field on $\mathfrak{M}^4$, and *it is these that we are interested in classifying* (up to homotopy).

For any trivial bundle $F\to X\xrightarrow{\Pi}B$, that is any bundle equivalent to the product bundle $F\overset{i}{\hookrightarrow}B\times F\xrightarrow{\pi}B$ where $i(x)=(b_0,x),\pi(b,x)=b$, the cross sections of Π are clearly in one-to-one correspondence with the set of continuous mappings $B\to F$. Two cross sections $C,C'\colon B\to X$ are said to be homotopic if there is a homotopy $C_t\colon B\to X$ with $C_0=C$, $C_1=C'$, and each C_t a cross section, $0\leqslant t\leqslant 1$. A lemma due to Barcus (1954) states that the requirement that each C_t be a cross section, $0<t<1$, may be dropped. We introduce the notation $[\![B,X]\!]$ for the set of homotopy classes of cross sections. For a trivial bundle the bijection between cross sections and maps $B\to F$ passes to a bijection $[\![B,X]\!]\approx[B,F]$, and it is this bijection that we will use. For a recent account of the use of fiber bundles in kink theories see Clarke (1979).

Remark. The bijection $\|B,X\|\approx[B,F]$ is not unique, but depends only on the explicit trivialization $X\approx B\times F$ of the bundle $F\to X\xrightarrow{\Pi} B$.

Definition 3.1. A space-time manifold $\mathfrak{M}^4$ is said to admit kinks if and only if $[\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!]$ contains at least two distinct elements.

From the above considerations we have

$$[\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!]\approx[\mathfrak{M}^4, S_{4,1}]$$

Now consider $S_{4,1}$. This space has the same homotopy type as $M_{4,1}=P^3$ (Steenrod, 1951, §40.8). Hence

$$[\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!]\approx[\mathfrak{M}^4, P^3]$$

Since $P^3=SO(3)$ (Steenrod, 1951, §22.3) is a topological group, $[\mathfrak{M}^4, P^3]$ is also a group (Spanier, 1966, p. 34). Thus $\mathfrak{M}^4$ admits kinks if and only if $[\mathfrak{M}^4, P^3]$ is not the trivial group.

A simple example is provided by $\mathfrak{M}^4=S^2\times R^1\times R^1$, with no boundary conditions on either R^1. Since R^1 is contractible,

$$[S^2\times R^1\times R^1, P^3]\approx[S^2, P^3]\approx\pi_2(P^3)\approx 0$$

Thus all metrics belong to the same homotopy class. A more interesting choice for $\mathfrak{M}^4$ would clearly give a less trivial answer for $[\mathfrak{M}^4, P^3]$.

Consider the example $\mathfrak{M}^4=S^3\times R^1$:

$$[S^3\times R^1, P^3]\approx[S^3, P^3]\approx\pi_3(P^3)\approx Z$$

Hence there is one type of kink with counting number $n\in Z$. This agrees with the results of Finkelstein and Misner (1959). The usual double covering $\kappa: S^3\to P^3$ gives rise to a 1-kink metric. The corresponding map $g: S^3\times R^1\to S_{4,1}$ is given by

$$g_{\mu\nu}(\phi,t)=\delta_{\mu\nu}-2\phi_\mu\phi_\nu, \qquad \mu,\nu=1,2,3,4$$

where $\delta_{\mu\nu}$ is the Kronecker delta, $t\in R^1$ and $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)\in S^3$ so that $\Sigma\phi_i^2=1$. Metrics of this type have been studied by Williams and Zia (1973), and Finkelstein and McCollum (1975). Note that the metric is actually given by

$$\mathbf{g}(\phi,t)=\sum g_{\mu\nu}(\phi,t)\boldsymbol{\theta}^\mu\otimes\boldsymbol{\theta}^\nu$$

where the θ's come from some explicit framing (i.e., an explicit trivialization of the tangent bundle of $\mathfrak{M}^4 = S^3 \times R^1$), which in turn defines the bijection $[\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!] \approx [\mathfrak{M}^4, S_{4,1}]$.

To better understand the relationship between $S_{4,1}$ and P^3 we shall give the explicit definition of the retraction $\pi: S_{4,1} \to P^3$ (Steenrod, 1951, §40.7). In fact π is a bundle map with contractible fiber equal to the product of the space of 3×3 positive definite matrices and the space of 1×1 negative definite matrices.

Let $\tau \in S_{4,1}$. Then $\tau\tau^T$ is positive definite so that $\alpha = (\tau\tau^T)^{1/2}$ exists, α also positive definite (and hence unique). One may then construct $\sigma = \tau\alpha^{-1}$ and it can be shown (Steenrod, 1951, §40) that $\sigma \in S_{4,1} \cap O(4)$. It follows from standard matrix theory of quadratic forms that one may find a (nonunique) $\rho \in O(4)$ such that $\sigma = \rho^T \sigma_1 \rho$, where $\sigma_1 = \mathrm{diag}(1,1,1,-1)$. Although ρ is not unique, its bottom row will be unique up to $\pm$ sign. Denote the bottom row of ρ by ρ_4 ($\rho_4 \in S^3$) and let square brackets [ ] denote the identification of antipodal points. The projection π is then defined by

$$\pi(\tau) = [\rho_4] \in P^3$$

For the example of $g_{\mu\nu}$ given above, it is easy to show that $\| g_{\mu\nu} \| = \rho^T \sigma_1 \rho$, where

$$\rho = \begin{pmatrix} \phi_4 & -\phi_3 & \phi_2 & -\phi_1 \\ \phi_3 & \phi_4 & -\phi_1 & -\phi_2 \\ -\phi_2 & \phi_1 & \phi_4 & -\phi_3 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{pmatrix} \in O(4)$$

and so $\pi(\| g_{\mu\nu} \|) = [\phi_1, \phi_2, \phi_3, \phi_4]$. Thus if we define a map $\gamma: S^3 \to S_{4,1}$ by

$$\gamma(\phi_1, \phi_2, \phi_3, \phi_4) = \| \delta_{\mu\nu} - 2\phi_\mu \phi_\nu \|$$

it follows that $\pi\gamma = \kappa: S^3 \to P^3$.

Now consider the problem of spin. In defining half-odd-integer spin in the general relativistic case, we cannot use R_t to rotate the whole of the manifold $\mathfrak{M}^4$, as was done with R^3. Instead we consider any point $p \in \mathfrak{M}^4$ and the tangent space T_p at that point. Let the frame $\{\mathbf{f}_\mu\}$ at p be a basis for T_p. A new basis can be obtained from the original one by $f_\mu'^\alpha = L^\alpha_\beta f^\beta_\mu$ where the matrix L belongs to the proper orthochronous Lorentz group $\mathfrak{L}^{+\uparrow}$. It is known that $\pi_1(\mathfrak{L}^{+\uparrow}) \approx Z_2$, the double connectedness being due to the $SO(3)$ subgroup of $\mathfrak{L}^{+\uparrow}$. If I_1 denotes the one-dimensional unit matrix then the direct sum $R_t \oplus I_1$ represents a nontrivial 2π rotation loop in $\mathfrak{L}^{+\uparrow}$.

Acting with L on $\{\mathbf{f}_\mu\}$ corresponds to acting with L^{-1} on $\{\boldsymbol{\theta}^\mu\}$, which in turn corresponds to acting on $\| g_{\mu\nu}\|$ according to $\| g_{\mu\nu}\| \to L^T \| g_{\mu\nu}\| L$.

Lorentz metric tensor fields are classified homotopically by $[\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!] \approx [\mathfrak{M}^4, S_{4,1}] \approx [\mathfrak{M}^4, P^3]$. Let $Q_w \in [\![\mathfrak{M}^4, T_{2L}(\mathfrak{M}^4)]\!]$ be any homotopy class. Denote the corresponding class in $[\mathfrak{M}^4, S_{4,1}]$ by Q_w^S, and the corresponding class in $[\mathfrak{M}^4, P^3]$ by Q_w^P. Consider a mapping $\gamma: \mathfrak{M}^4 \to S_{4,1}$ and let $\gamma \in Q_w^S$. A simple example of what we shall mean by a 2π rotation path in Q_w^S is given by $\rho: I^1 \to Q_w^S$ with

$$\rho(t)(p) = (R_t \oplus I_1)^T \gamma(p)(R_t \oplus I_1)$$

where $p \in \mathfrak{M}^4$ and $t \in I^1$.

Definition 3.2. If the 2π rotation loops in Q_w^S are nontrivial, then Q_w^S (and also Q_w) is said to admit half-odd-integer spin.

Since P^3 is homeomorphic to $SO(3)$ we can define a 2π rotation loop in Q_w^P by allowing R_t to act directly on P^3 by the usual group operation. It turns out (see Section 5) that this path corresponds directly to the 2π rotation loop in Q_w^S under the projection $\pi: S_{4,1} \to P^3$. Thus the 2π rotation loops in Q_w, Q_w^S, and Q_w^P are either all trivial or all nontrivial. For analyzing problems of spin we can use Q_w, Q_w^S, and Q_w^P interchangeably, and we shall often omit the superscripts S and P and refer simply to Q_w.

For proving certain theorems in general relativity it is sometimes assumed that there is a fibration of $\mathfrak{M}^4$ by a system of time-lines (Lichnerowicz, 1968, p. 110). Assuming that there are no closed timelike curves, this means that $\mathfrak{M}^4$ would be a vector bundle space whose fiber is R^1 and whose base is a 3-manifold M. For such a fibration it is standard that M (identified as the zero-section of the bundle) is a strong deformation retract of $\mathfrak{M}^4$. It follows that $[\mathfrak{M}^4, Y] \approx [M, Y]$ for any space Y. A particularly simple example of an $\mathfrak{M}^4$ of the above type would be any space-time manifold of the form $M \times R^1$. We shall always assume M to be closed, connected, and orientable.

This paper takes the point of view that a general relativistic kink can serve as a model for an elementary particle. Thus the structure M_0 of interest should be localized within a 2-sphere S (Figure 1 illustrates the situation in one less dimension). At S space is Euclidean and in terms of mappings into P^3 all points on S are mapped into the group identity. The classification problem is then equivalent to considering maps of the quotient space $M = M_0/S$. Note that M is a closed, connected manifold.

Before proceeding further it is useful to first mention some conventions concerning notation and base points that will be used henceforth. As base point of $S_{4,1}$ we take $\sigma_1 = \mathrm{diag}(1,1,1,-1)$. As base point of any of the

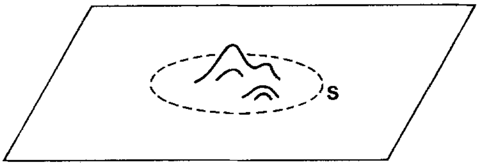

Fig. 1. 2-sphere bounding homotopically nontrivial region.

various Lie groups that occur we take the identity element. For example regarding S^3 as the unit quaternions, the base point of S^3 is $I=1+0i+0j+0k$. We regard P^3 as S^3 with antipodal points identified, and write $[x]\in P^3$ for the equivalence class containing $x, -x$. It is also convenient to use column vectors for points of S^3 and P^3. In this notation $a+bi+cj+dk=\mathrm{col}(b,c,d,a)\in S^3$ and $[a+bi+cj+dk]=\mathrm{col}[b,c,d,a]\in P^3$. Note that $\mathrm{col}(0,0,0,1)$ and $\mathrm{col}[0,0,0,1]$ are the respective base points. As mentioned in Section 1, only base-point-preserving maps are considered.

4. CLASSIFICATION OF RELATIVISTIC KINKS

Throughout Sections 4 and 5 we shall assume that the space-time manifold $\mathfrak{M}^4$ is a bundle space whose fiber is R^1 and whose base is a closed, connected, orientable 3-manifold M. Since $[\mathfrak{M}^4, P^3]\approx[M, P^3]$, it follows that the set of homotopy equivalence classes of metrics will be isomorphic to $[M, P^3]$.

We first show that $[M, S^3]\approx Z$. The situation for $[M, P^3]$ is more complicated, although one knows at least that this is a group (cf. Section 3). In Theorem 4.3 we show that $[M, P^3]$ is Abelian. The main result of this section follows from Theorem 4.3 and is summarized by Theorem 4.6, which asserts that

$$[M, P^3]\approx Z\oplus Z_2^l$$

where Z_2^l denotes $Z_2\oplus\cdots\oplus Z_2$ (l copies). The value of l is determined from $H^1(M;Z_2)$. Since the latter is a finite-dimensional vector space over Z_2, we have $H^1(M;Z_2)\approx Z_2^k$ for some $k\geqslant 0$. Then either $l=k$ or $l=k-1$. When $l=k$, we will call M a type 2 manifold, and when $l=k-1$, M will be called type 1. Theorem 4.4 shows that M is type 1 if and only if M admits a degree-1 map into P^3, and this theorem also gives methods to determine whether a given 3-manifold is type 1 or type 2. For the rest of this section some familiarity with various techniques of homotopy theory (cf. Mosher and Tangora, 1968) is assumed. We shall use $\hookrightarrow$ to denote inclusion, $\rightarrowtail$ to denote monic (one-to-one), and $\twoheadrightarrow$ to denote epic (onto).

To classify maps of any connected CW complex X, $\dim X \leqslant 4$, into S^3, a 2-stage Postnikov system for S^3 is used:

$$\begin{tikzcd} K(Z_2,4) \arrow[r, hook] & X_4 \arrow[d] & \\ & K(Z,3) \arrow[r, "\mathrm{Sq}^2"] & K(Z_2,5) \end{tikzcd}$$

The space X_4 is defined by the Postnikov system construction, and Sq^2 is the Steenrod squaring operation. Under these assumptions $[X,S^3]\approx[X,X_4]$.

Proposition 4.1. $[M,S^3]\approx Z$ for any closed, connected, orientable 3-manifold M.

Proof. Immediate from the fiber mapping sequence:

$$\begin{tikzcd} {[M,K(Z_2,4)]} \arrow[r] \arrow[d, phantom, "\wr"] & {[M,X_4]} \arrow[r] \arrow[d, phantom, "\wr"] & {[M,K(Z,3)]} \arrow[r] \arrow[d, phantom, "\wr"] & {[M,K(Z_2,5)]} \arrow[d, phantom, "\wr"] \\ H^4(M;Z_2)\approx 0 & {[M,S^3]} & H^3(M;Z)\approx Z & H^5(M;Z_2)\approx 0 \end{tikzcd}$$

■

Remark. This result can be proved using simpler tools such as the Hopf classification theorem, but it is convenient to introduce X_4 here since it will be needed later in Section 5.

As before let $\kappa: S^3 \to P^3$ be the standard double covering identifying antipodal points. Let $P^\infty = P^1 \cup P^2 \cup \cdots$ and let $\mu: P^3 \hookrightarrow P^\infty$ be the inclusion. Since κ is a homomorphism of topological groups, the mapping $\kappa_\#$, where

$$\kappa_\#: [X,S^3] \to [X,P^3]$$

is also a group homomorphism. When X is simply connected $\kappa_\#$ is an isomorphism (Hu, 1959, p. 97) and $[X,P^3]\approx Z$. However, $X=S^3$ is the only known example of a simply connected closed 3-manifold. In general, $[X,P^3]$ is somewhat more complicated as we see in the following result.

Proposition 4.2. For any pathwise connected space X there is a short exact sequence of groups

$$[X,S^3] \overset{\kappa_\#}{\rightarrowtail} [X,P^3] \overset{\mu_\#}{\twoheadrightarrow} [X,P^\infty] \approx H^1(X;Z_2)$$

Proof. Consider the commutative diagram of fibrations

$$
\begin{tikzcd}
Z_2 \arrow[r, phantom, "\approx"] \arrow[d, hook] & Z_2 \arrow[r, phantom, "\approx"] \arrow[d, hook] & \Omega K(Z_2,1) \arrow[d, hook] \\
S^3 \arrow[r, hook] \arrow[d, "\kappa"] & S^\infty \arrow[r, phantom, "\approx"] \arrow[d, "\kappa_\infty"] & PK(Z_2,1) \arrow[d] \\
P^3 \arrow[r, hook, "\mu"] & P^\infty \arrow[r, phantom, "\approx"] & K(Z_2,1)
\end{tikzcd}
$$

where $PY, \Omega Y$ are, respectively, the path space and loop space of a space Y with base point y_0. We may therefore regard κ as the induced fibration of the standard path fibration over P^∞. The fiber mapping sequence together with $[X, Z_2] \approx 0$ now gives exactness.

It must still be shown that $\mu_\#$ is a homomorphism. (Note that μ is not a group homomorphism; indeed P^∞ is an H-space but not a topological group). Suppose $f, g: X \to P^3$, and let $m: P^3 \times P^3 \to P^3$ be the multiplication on P^3. Then $[f]\cdot[g]$ is represented by the composition $h = m \circ (f \times g) \circ \Delta$:

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} P^3 \times P^3 \xrightarrow{m} P^3$$

Letting ξ generate $H^1(P^n; Z_2) \approx Z_2$, $n \geqslant 2$, it follows by definition of $\mu_\#$ that $\mu_\#[h] = h^*(\xi)$. But $\mu_\#[h] = \mu_\#([f]\cdot[g])$, and

$$
\begin{aligned}
h^*(\xi) &= \Delta^*(f \times g)^* m^*(\xi) \\
&= \Delta^*(f^* \otimes g^*)(\xi \otimes I + I \otimes \xi) \\
&= \Delta^*(f^*\xi \otimes I + I \otimes g^*\xi) \\
&= f^*\xi + g^*\xi \\
&= \mu_\#[f] + \mu_\#[g]
\end{aligned}
$$

Hence $\mu_\#$ is a homomorphism. ∎

Corollary. For any connected CW complex X with $\dim X \leqslant 3$ there is an exact sequence of groups

$$[X, S^3] \overset{\kappa_\#}{\rightarrowtail} [X, P^3] \overset{\mu_\#}{\twoheadrightarrow} H^1(X; Z_2)$$

Proof. $\mu_\#$ is now necessarily onto by the cellular approximation theorem (Mosher and Tangora, 1968, p. 129) since $P^3 = (P^\infty)^{(3)}$. (We use $X^{(3)}$ to denote the 3-skeleton of a cellular space X, i.e., the subspace formed by all cells of dimension less than or equal to 3.) ∎

Theorem 4.3. For any closed orientable 3-manifold M there is a short exact sequence $\mathfrak{S}$ of Abelian groups

$$\mathfrak{S}: 0 \to Z \xrightarrow{\kappa_\#} [M, P^3] \xrightarrow{\mu_\#} H^1(M; Z_2) \to 0$$

Proof. Using the above corollary and Proposition 4.1 gives all the stated conclusions except that the commutativity of $G = [M, P^3]$ must still be demonstrated. The following proof of this fact simplifies the authors' original proof and was suggested by Prof. P. Hilton.

Let $P^3 \xrightarrow{\rho} S^3$ be the pinch map. The composition $Z = [M, S^3] \xrightarrow{\kappa_\#} [M, P^3] \xrightarrow{\rho_\#} [M, S^3] = Z$ is then multiplied by 2. Let $a = \kappa_\#(1)$, which generates Ker $\mu_\#$. Take any $b, c \in [M, P^3]$ and let $d = bcb^{-1}c^{-1}$. Since $H^1(M; Z_2)$ is Abelian $\mu_\#(d) = 0$, so $d = a^m$ for some m. Since Z is Abelian $\rho_\#(d) = 0$. Thus $0 = \rho_\#(a^m) = \rho_\#\kappa_\#(m) = 2m$, and $m = 0, d = 1, bc = cb$. ∎

Remark. Only the commutativity of $H^1(M; Z_2)$ was used in this proof.

A short exact sequence such as

$$0 \to A \to B \to C \to 0$$

is said to *split* if it is isomorphic by the obvious commutative diagram to the short exact sequence

$$0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \to 0$$

where $\iota(a) = (a, 0)$ and $\pi(a, c) = c$. The exact sequence $\mathfrak{S}$ of Theorem 4.3 may or may not split. For example, it is know that $[P^3, P^3] \approx Z$ (Milgram and Zvengrowski, 1974, Lemma 2.1) so in this case $\mathfrak{S}$ does not split. Clearly, $[M, P^3] \approx Z \oplus Z_2^k$ when $\mathfrak{S}$ splits. When $\mathfrak{S}$ does not split, it is not hard to see (using Theorem 4.4 below or otherwise) that $\mathfrak{S}$ is isomorphic to the direct sum of the simpler sequences:

$$\mathfrak{S}_1: 0 \to Z \xrightarrow{\times 2} Z \to Z_2 \to 0$$

$$\mathfrak{S}_2: 0 \to 0 \to Z_2^{k-1} \xrightarrow{\approx} Z_2^{k-1} \to 0$$

whence $[M, P^3] \approx Z \oplus Z_2^{k-1}$. Letting $l = k$ or $k - 1$ as necessary, it follows that any map $\varphi: M \to P^3$ has a *kink-type* $(n; t_1, \ldots, t_l)$ where $n \in Z, t_i \in Z_2$, $i = 1, 2, \ldots, l$. Note that under different choices of generators for $[M, P^3]$ the t_i may change but n is unique (up to a sign). We shall call n the *kink* *number* of the map φ (and refer to φ as an *n-kink map*). Let us say that M is of type 1 if $\mathfrak{S}$ does not split and type 2 if $\mathfrak{S}$ splits. The next theorem gives criteria to determine the type of M.

Theorem 4.4. The following statements are equivalent:

(i) M is of type 1.
(ii) $\kappa_{\#}(J)$ is divisible by 2 in $[M,P^3]$.
(iii) M admits a degree-1 map into P^3 (hence the term "type 1").
(iv) There is an element $\xi \in H^1(M;Z_2)$ such that $0 \neq \xi^3 \in H^3(M;Z_2)$.

Proof. (i)$\Leftrightarrow$(ii) is obvious, since type 1 means that $\mathfrak{S}$ does not split, which is clearly equivalent to $\kappa_{\#}(J)$ being divisible by 2 in $[M,P^3]$ since Coker $\kappa_{\#}$ is a finite direct sum of Z_2's.

(ii)$\Rightarrow$(iii). Let the generator $J \in [M,S^3]$ be represented by a degree-1 map g. Then $\kappa g \simeq h$ where $[h] \in [M,P^3]$ is divisible by 2. (The symbol $\simeq$ denotes "homotopic to"). This means $h \simeq m\circ(f\times f)\circ\Delta$ as in the proof of Proposition 4.2, for some $f\colon M\to P^3$. Now

$$2 = \deg\kappa \cdot \deg g = \deg(\kappa g) = \deg h = \deg f + \deg f$$

Hence $\deg f = 1$.

(iii)$\Rightarrow$(ii). Let $f\colon M\to P^3, \deg f = 1$, and take $h = m\circ(f\times f)\circ\Delta$ as above. Since $\pi_1(P^3) \approx Z_2$,

$$0 = h_*\colon \pi_1(M) \to \pi_1(P^3)$$

Hence there is a mapping $g\colon M\to S^3$ such that $\kappa g = h$. As above one readily finds $\deg g = 1$, hence $[g]$ generates $[M,S^3]$ and $\kappa_{\#}(J) = \pm\kappa_{\#}[g] = \pm[\kappa\circ g] = \pm[h] = \pm 2[f]$ is divisible by 2.

(iii)$\Rightarrow$(iv). Let η generate $H^1(P^3,Z_2)$. Then $\eta^3 \neq 0$. Any degree-1 map $f\colon M\to P^3$ will have $f^*\colon H^3(P^3;Z_2)\to H^3(M;Z_2)$ an isomorphism. (f^* is "multiplication by 1"). Taking $\xi = f^*(\eta)$ gives $\xi^3 = f^*(\eta^3) \neq 0$.

(iv)$\Rightarrow$(iii). Let $\xi \in H^1(M;Z_2)$, $\xi^3 \neq 0$, and write ι_1 for the generator of $H^1(P^\infty;Z_2)$. Since P^∞ is a $K(Z_2,1)$, it follows that there is a map $g\colon M\to P^\infty$ with $g^*(\iota_1) = \xi$. By the cellular approximation theorem $g \simeq if$ for some map $f\colon M\to P^3$, where $i\colon P^3 \hookrightarrow P^\infty$ is the inclusion. Since $i^*(\iota_1) = \eta$, one has $f^*(\eta) = \xi$. Hence $f^*(\eta^3) = \xi^3 \neq 0$, which means that f is a map of odd degree, say degree $2k+1$. Finally, let $f_1\colon M\to P^3$ have degree $2k$ (obtainable by a composition of κ with any degree k map $M\to S^3$). Then any map representing $[f]-[f_1]$ will have degree 1. This completes the proof of the theorem. ∎

In many cases the following corollary is useful.

Corollary 4.5. If M has type 1 then Z_2 is a direct summand of $H_1(M)$.

Proof. By the above theorem there is a map $f\colon M \to P^3$ of degree 1. But then $f_*\colon H_1(M) \to H_1(P^3) \approx Z_2$ is a split epimorphism (Siebenmann, 1969, §2.11), and hence Z_2 is a direct summand of $H_1(M)$. ∎

The main results so far can be summarized in the following theorem.

Theorem 4.6. (a) M has type 1 if and only if there exists $\xi \in H^1(M; Z_2)$ such that $\xi^3 \neq 0$. Otherwise M has type 2. (b) Letting $H^1(M; Z_2) \approx Z_2^k$ one has

$$[M, P^3] \approx Z \oplus Z_2^l$$

where

$$l = \begin{cases} k-1 & \text{if } M \text{ has type 1} \\ k & \text{if } M \text{ has type 2} \end{cases}$$

Remark. It would be interesting to see whether the type-1–type-2 dichotomy has a corresponding physical significance.

5. SPIN

In Section 3 the 2π rotation loop ρ of a map $\gamma\colon \mathfrak{M}^4 \to S_{4,1}$ was defined by

$$\rho(t)(p) = (R_t \oplus I_1)^T \gamma(p)(R_t \oplus I_1)$$

where $t \in [0,1]$, $p \in \mathfrak{M}^4$. Under the fiber map $\pi\colon S_{4,1} \to M_{4,1} = P^3 = SO(3)$ (Steenrod, 1951, §40.7) this induces a loop λ in $[\mathfrak{M}^4, SO(3)]$. According to Steenrod (1951, §40.8 and an observation concluding §40.2), this induced loop is given by

$$\lambda(t)(p) = R_t \cdot \pi\gamma(p)$$

In other words the conjugation action by orthogonal matrices in $S_{4,1}$ corresponds to the left multiplication action in $M_{4,1} = SO(3)$, under the fibration π. Again following Section 3, we may replace $\mathfrak{M}^4$ in our computation by a closed, connected, orientable 3-manifold M.

Just as the kinks were classified by $[M, P^3]$, we shall see below that the 2π rotation loops are classified as elements of $[\Sigma M, P^3]$ where ΣM is the (reduced) suspension of M. The definition of suspension can be found in the book by Spanier (1966, p. 41). In Theorem 5.2 we compute this group. The main result on spin is proved in Theorem 5.5, which asserts that for suitable maps $\varphi: M \to P^3$ of kink number ± 1 (or $2m+1$) the corresponding 2π rotation loop λ represents a nonzero element in $[\Sigma M, P^3]$ of order 2. The exact nature of this element depends on whether M has type 1 or type 2.

In accordance with the above observations we first make a definition.

Definition 5.1. Let $\varphi: M \to P^3 = SO(3)$, and denote its component in the space $(M, P^3)_*$ of based maps by Q_φ (or Q_w, if w is the kink type of φ). Then λ_φ is defined to be the 2π rotation loop in Q_w given by

$$\lambda_\varphi(t)(x) = R_t \cdot \varphi(x)$$

where $t \in [0,1]$, $x \in M$.

Remark. In the special case $\varphi = \kappa: S^3 \to P^3$, the loop λ_κ is precisely that studied by Williams and Zvengrowski (1977), and is known to to be nontrivial of order 2.

Suppose now that $\varphi \in Q_w$, as above. Let Q_0 be the component of $(M, P^3)_*$ containing the constant map φ_0. Using φ we can define a "translation" homeomorphism $\tau: Q_w \to Q_0$, by

$$\tau(\psi)(x) = [\varphi(x)]^{-1} \cdot \psi(x)$$

where $x \in M, \psi \in Q_w$. Here we have taken advantage of the product in P^3. This now gives rise to a loop λ_0 in Q_0, where $\lambda_0 = \tau \circ \lambda_\varphi$. The "adjoint" $\tilde{\varphi}$ of λ_0 is defined by

$$\tilde{\varphi}(x,t) = \lambda_0(t)(x)$$

and it is easy to check that $\tilde{\varphi}$ is a well-defined map $\tilde{\varphi}: \Sigma M \to P^3$. Before passing to our main theorems we give an explicit example which may help to illustrate this process.

Example. Take $M = P^3, \varphi = id_{P^3}$. Denote points of P^3 by $x = \mathrm{col}[x_1, x_2, x_3, x_4] = \mathrm{col}[-x_1, -x_2, -x_3, -x_4], x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, and write $\theta = 2\pi t$. After some calculations, one finds

$$\lambda_\varphi(t)(x) = \begin{bmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \\ x_3 \\ x_4 \end{bmatrix}$$

$$\tilde{\varphi}(x,t) = \begin{bmatrix} (x_2x_4 - x_1x_3)\sin\theta - (x_1x_4 + x_2x_3)(1-\cos\theta) \\ -(x_1x_4 + x_2x_3)\sin\theta + (x_1x_3 - x_2x_4)(1-\cos\theta) \\ (x_1^2 + x_2^2)\sin\theta \\ x_3^2 + x_4^2 + (x_1^2 + x_2^2)\cos\theta \end{bmatrix}$$

Note that $\tilde{\varphi}(x,0)=\tilde{\varphi}(x,1)=\tilde{\varphi}(I,t)=I$, where $I=\mathrm{col}[0,0,0,1]$ is the base point of P^3, and hence $\tilde{\varphi}$ is indeed a map on ΣP^3.

Theorem 5.2. $[\Sigma M, P^3]\approx Z_2\oplus H_1(M)$.

Proof. Write $Y=\Sigma M$. Since Y is simply connected, it follows (Hu, 1959, p. 97) that

$$[Y,P^3]\approx[Y,S^3]$$

Using the fiber mapping sequence applied to the two-stage Postnikov system of Section 4, we obtain an exact sequence:

$$\begin{tikzcd}
H^2(Y) \arrow[r, "\mathrm{Sq}^2"] \arrow[d, phantom, "\wr"] & H^4(Y;Z_2) \arrow[r, "i_*"] \arrow[d, phantom, "\wr"] & {[Y,X_4]} \arrow[r, "\pi_*"] & H^3(Y) \arrow[r] \arrow[d, phantom, "\wr"] & H^5(Y;Z_2) \arrow[d, phantom, "\wr"] \\
H^1(M) \arrow[r, "\mathrm{Sq}^2"] & H^3(M;Z_2)\approx Z_2 & & H^2(M) \arrow[d, phantom, "\wr"] & 0 \\
 & & {[Y,S^3]} \arrow[uu, "\rho_*", "\approx"'] & H_1(M) & \text{(Poincaré duality)}
\end{tikzcd}$$

The left-hand square commutes since Sq^2 commutes with suspension. However, Sq^i vanishes on any class of dimension less than i, hence $\mathrm{Sq}^2=0$ in the diagram and i_* is monic, π_* epic. It remains to prove that the resulting short exact sequence splits.

To do this, let u generate $H^4(Y;Z_2)\approx Z_2$, and set $i_*(u)=\rho_*[u']$ for some $u'\colon Y\to S^3$. We must simply show $[u']\neq 2[v]$ for any $[v]\in[Y,S^3]$ in order to establish the splitting. Denoting stable homotopy classes by $\{\,,\}$, we have $[Y,S^3]\approx\{Y,S^3\}$ (Spanier, 1966, p. 458) and therefore techniques of stable homotopy theory are applicable. Consider the cofibration sequence

$$S^3\xrightarrow{u}S^3\overset{j}{\hookrightarrow}P_3^4\to S^4\to\cdots$$

where u is a degree-2 map (note P_k^n is defined as P^n/P^{k-1}). This gives rise to an exact sequence

$$\{Y,S^3\}\xrightarrow{u_*}\{Y,S^3\}\xrightarrow{j_*}\{Y,P_3^4\}\to\cdots$$

with u_* being multiplication by 2. Thus $[u']\neq 2[v]$ if and only if $j_*[u']\neq 0$.

Next consider u as a map $u\colon Y\to K(Z_2,4)$, which factors by cellular approximation into $Y\xrightarrow{\omega}S^4\hookrightarrow K(Z_2,4)$. Taking ν as the (suspended) Hopf map $S^4 \to S^3$, the triangle below is homotopy commutative:

$$\begin{tikzcd} & & S^3 \arrow[r, hook, "j"] & P_3^4 \arrow[d] \\ Y \arrow[r, "\omega"] \arrow[rr, bend right, "u"'] & S^4 \arrow[ur, "\nu"] \arrow[r, hook] & K(Z_2,4) \arrow[r, hook, "i"] & X_4 \end{tikzcd}$$

Then $\rho_*[\nu\omega]+[\rho\nu\omega]=[iu]=i_*[u]=\rho_*[u']$ gives $\nu\omega \simeq u'$, again since ρ_* is an isomorphism.

Letting $j\nu = f$, one verifies $C_f = P_3^5$, where C_f denotes the mapping cone of f. There is then a commutative diagram of cofibration sequences:

$$\begin{tikzcd} Y \arrow[r, "f\omega"] \arrow[d, "\omega"'] & P_3^4 \arrow[r, hook] \arrow[d, "\shortparallel"] & C_{f\omega} \arrow[r] \arrow[d] & \Sigma Y \arrow[r] \arrow[d, "\Sigma\omega"] & \cdots \\ S^4 \arrow[r, "f"] & P_3^4 \arrow[r, hook] & P_3^5 \arrow[r] & S^5 \arrow[r] & \cdots \end{tikzcd}$$

A functional operation Sq_f^2 is then defined on $\xi_3 \in H^3(P_3^4; Z_2)$, and one finds from the known action of Sq^2 in P_3^5 that

$$\mathrm{Sq}_f^2(\xi_3) = \frac{3!}{1!2!}\mu_4 = 3\mu_4 = \mu_4$$

where μ_4 generates $H^4(S^4; Z_2)$, modulo zero indeterminacy. By naturality $\mathrm{Sq}_{f\omega}^2(\xi_3) = \omega^*(u_4) = u \neq 0$, again mod 0. Hence $f\omega$ is nontrivial. But $f\omega = j\nu\omega \simeq ju'$, so $j_*[u'] = [ju'] \neq 0$, as was required to be proved.

Corollary 5.3. $[\Sigma P^3, P^3] \approx Z_2 \oplus H_1(P^3) = Z_2 \oplus Z_2$.

Lemma 5.4. $[\widetilde{\mathrm{id}}_{P^3}] = (\lambda, \zeta) \in [\Sigma P^3, P^3]$, where ζ is the generator of $H_1(P^3)$ and $\lambda \in Z_2$ is undetermined.

Proof. $\Sigma\kappa: S^4 \to \Sigma P^3$, so

$$(\Sigma\kappa)^{\#}: [\Sigma P^3, P^3] \to [S^4, P^3]$$

and clearly $(\Sigma\kappa)^{\#}[\widetilde{\mathrm{id}}] = [\tilde{\kappa}]$. As mentioned above in the remark following Definition 5.1, λ_κ or equivalently its adjoint $\tilde{\kappa}$ has been studied by Williams and Zvengrowski (1977), and it was shown there that $[\tilde{\kappa}]$ is the nonzero element of $[S^4, P^3] \approx \pi_4(P^3) \approx Z_2$. The cofibration sequence

$$S^3 \xrightarrow{\kappa} P^3 \xhookrightarrow{i} P^4 \xrightarrow{\rho} S^4 \xrightarrow{\Sigma\kappa} \Sigma P^3$$

taken together with the usual 2-stage Postnikov system for S^3 now gives a commutative diagram:

$$
\begin{tikzcd}
H^2(P^4) \arrow[r, "Sq^2", "\approx"'] & H^4(P^4;Z_2) \arrow[r] & {[P^4,S^3]} \arrow[r] & H^3(P^4)\approx 0 \\
& H^4(S^4;Z_2) \arrow[u, "\rho^*"] \arrow[r, "\approx"] & {[S^4,S^3]} \arrow[u, "\rho^\#"] & \\
& Z_2\approx H^4(\Sigma P^3;Z_2) \arrow[u, "(\Sigma\kappa)^*"] \arrow[r, tail] & {[\Sigma P^3,S^3]} \arrow[u, "(\Sigma\kappa)^\#"] \arrow[r, two heads] & H^3(\Sigma P^3) \\
& & Z_2\oplus Z_2 \arrow[u, phantom, "\wr"] & H_1(P^3)\approx Z_2 \arrow[u, phantom, "\wr"]
\end{tikzcd}
$$

The top row shows $[P^4,S^3]=0$, hence $(\Sigma\kappa)^\#$ is epic. Now κ has degree 2, hence $(\Sigma\kappa)^*=0$. It follows that $(\Sigma\kappa)^\#$ must be nonzero on $(0,\zeta)\in Z_2\oplus Z_2$ and is zero on $(1,0)$. But we have remarked that $(\Sigma\kappa)^\#[\widetilde{\mathrm{id}}]=[\tilde\kappa]\neq 0$, so $[\widetilde{\mathrm{id}}]=(0,\zeta)$ or $(1,\zeta)$ are the only possibilities.

Theorem 5.5. If $\varphi\colon M\to P^3$ is a map of kink number ± 1 and kink type $(\pm 1;0,0,\ldots,0)$ in case M is of type 2, then $[\tilde\varphi]$ is nonzero of order 2 in $[\Sigma M,P^3]$.

Proof. First suppose M has type 2, in which case the 1-kink map $\varphi\colon M\to P^3$ is a composition $M\xrightarrow{g} S^3 \overset{\kappa}{\twoheadrightarrow} \rho^3$, where $\deg g=1$. Let $\iota=\mathrm{id}_{S^3}$. Now $\tilde\iota\in[\Sigma S^3,S^3]\approx\pi_4(S^3)\approx Z_2$ was shown (Williams and Zvengrowski, 1977) to be the nonzero element, and it is clear from the definitions of 2π rotations that $g^*(\lambda_\iota)=\lambda_g$ or equivalently $(\Sigma g)^\#[\tilde\iota]=[\tilde g]$. Consider the commutative diagram (cf. first exact sequence in proof of Theorem 5.2):

$$
\begin{tikzcd}
Z_2\approx H^4(\Sigma M;Z_2) \arrow[r, "i_*"] & {[\Sigma M,S^3]} \\
Z_2\approx H^4(S^4;Z_2) \arrow[u, "g^*"] \arrow[r, "\approx"] & {[S^4,S^3]} \arrow[u, "(\Sigma g)^\#"']
\end{tikzcd}
$$

Then $[\tilde g]=(\Sigma g)^\#[\tilde\iota]=i_*g^*(u_4)$, where u_4 generates $H^4(S^4;Z_2)$. But $\deg g=1$, so g^* is an isomorphism and hence $[\tilde g]$ has order 2. Applying the isomorphism $\kappa_\#\colon[\Sigma M,S^3]\to[\Sigma M,P^3]$, it follows that $[\tilde\varphi]$ also has order 2.

Now suppose M is of type 1, in which case $\deg\varphi = 1$. Consider

$$\begin{tikzcd}
Z_2 \approx H^4(\Sigma M; Z_2) \arrow[r, "i"] & {[\Sigma M, S^3]} \arrow[r, two heads] & H^2(M) \\
Z_2 \approx H^4(\Sigma P^3; Z_2) \arrow[u, "(\Sigma\varphi)^*"'] \arrow[r, tail] & {[\Sigma P^3, S^3]} \arrow[u, "(\Sigma\varphi)^\#"'] \arrow[r, two heads] \arrow[d, phantom, "\wr"] & H^2(P^3) \arrow[u, "\varphi^*"'] \arrow[d, phantom, "\wr"] \\
& Z_2 \oplus Z_2 & H_1(P^3) \approx Z_2
\end{tikzcd}$$

Here $[\tilde{\varphi}] = (\Sigma\varphi)^{\#}[\mathrm{id}_{P^3}] = (\Sigma\varphi)^{\#}(\lambda, \zeta)$ by Lemma 5.4.

Now $\deg\varphi = 1$ implies $(\Sigma\varphi)^*$ is an isomorphism and φ^* monic. Hence $[\tilde{\varphi}] = (\lambda, \varphi^*\xi_2)$, where ξ_2 generates $H^2(P^3) \approx Z_2$, and this class is of order 2 since φ^* is monic. ■

Remark. Note that $[\tilde{\varphi}]$, although always of order 2, has a somewhat different form in the type 1 and type 2 cases.

Remark. Exactly the same proof works for maps of odd kink number.

Remark. Suppose we were interested in a Skyrme-type theory where we were mapping into S^3, and where the domain was any closed, connected, orientable 3-manifold M. The fact that $[M, S^3] \approx Z$ (Proposition 4.1) means that such a theory admits kinks, and the kink number is equal to the degree. It is also easy to show (following the first half of the proof of Theorem 5.5) that the homotopy classes $Q_n \in [M, S^3]$ admit half-odd-integer spin if and only if n is odd.

6. EXAMPLES

Table I lists some orientable 3-manifolds of interest in relativity theory (Fisher, 1970, p. 334). The symbol # denotes the connected sum of two closed, orientable 3-manifolds, M_1 and M_2, formed by removing two open 3-balls from M_1 and M_2, respectively, and identifying the resulting 2-sphere boundaries by an orientation-reversing homeomorphism. $S(p)$ denotes the 3-sphere with p handles,

$$S(p) = S^3 \# (S^1 \times S^2)_1 \# \cdots \# (S^1 \times S^2)_p$$

We write $L(p,q)$ for the lens space of type (p,q). For any group $G \subset SO(3) = P^3, \bar{G} = \kappa^{-1}(G) \subset S^3$ denotes the "double cover" of G. By D_n we mean

TABLE I. Kink Classification for Some choices of M[a]

M		$\pi_1(M)$	$H_1(M)$	Type	Kinks$=[M,P^3]$
S^3		0	0	2	Z
P^3		Z_2	Z_2	1	Z
$S^1\times S^2$		Z	Z	2	$Z\oplus Z_2$
$S^1\times S^1\times S^1$		Z^3	Z^3	2	$Z\oplus Z_2^3$
$S(p)$		F_p	Z^p	2	$Z\oplus Z_2^p$
$L(p,q)$	$p\equiv 1 \pmod 2$	Z_p	Z_p	2	Z
	$p\equiv 2 \pmod 4$	Z_p	Z_p	1	Z
	$p\equiv 0 \pmod 4$	Z_p	Z_p	2	$Z\oplus Z_2$
$S^3/\bar{D}_n$	$n\equiv 1 \pmod 2$	$\bar{D}_n$	Z_4	2	$Z\oplus Z_2$
	$n\equiv 2 \pmod 4$	$\bar{D}_n$	Z_2^2	2	$Z\oplus Z_2^2$
	$n\equiv 0 \pmod 4$	$\bar{D}_n$	Z_2^2	1	$Z\oplus Z_2$
$S^3/\bar{T}$		$\bar{T}$	Z_3	2	Z
$S^3/\bar{O}$		$\bar{O}$	Z_2	1	Z
$S^3/\bar{I}$		$\bar{I}$	0	2	Z

[a] G^p denotes $G\oplus\cdots\oplus G$ (p times) for an Abelian group G, and F_p denotes the free group on p symbols.

the dihedral group of order $2n$, and T, O, I denote the tetrahedral, octahedral, and icosahedral groups of orders 12, 24, and 60, respectively.

The classification into type 1 or type 2 manifolds is based on Theorem 4.4 (iv) or on Corollary 4.5 in the simpler cases. The necessary cohomological calculations for manifolds of the form S^3/π, where π is a group acting freely on S^3 (e.g., $\bar{G}$ above), are related to the cohomology of the group π by Lemma 6.1 below, but further details of these calculations are omitted. The methods of Cartan and Eilenberg (1956, Chap. 12) can be used, or see also Shastri and Zvengrowski (1979).

Lemma 6.1. Let π be a group acting freely on S^n, Λ a ring of coefficients with trivial π action, and $M=S^n/\pi$. Suppose $\xi\in H^1(\pi;\Lambda)$ and $\xi^n\neq 0$. Then there is an element $\eta\in H^1(M;\Lambda)$ such that $\eta^n\neq 0$.

Proof. M can be taken as the n-skelton of a $K(\pi,1)$. Thus $X=K(\pi,1)$ $=M\cup e_\alpha^{n+1}\cup e_\beta^{n+2}\cup\cdots$. Let $i\colon M\hookrightarrow X$ be the inclusion. For any coefficients, i^* is an isomorphism in dimensions less than n and a monomorphism in dimension n. Since $H^*(\pi;\Lambda)=H^*(X;\Lambda)$ (by definition), taking $\eta=i^*(\xi)$ clearly suffices.

Corollary 6.2. With $n=3$ above and $\Lambda=Z_2$, M is then of type 1 (cf. Theorem 4.4).

Corollary 6.3. If $\pi = Z_2 \times H$ for some group H and $n=3$, then M is of type 1.

Proof. $K(\pi, 1) = P^\infty \times K(H, 1)$, so taking $\xi = \iota_1 \otimes 1$ with Z_2 coefficients gives the result. ■

Only irreducible manifolds (except for $S(p)$) have been considered in Table I, that is manifolds that cannot be expressed nontrivially as a connected sum. Note that a connected sum of 3-manifolds, $M_1 \# \cdots \# M_r$ will be of type 1 if any M_i is of type 1, and will be of type 2 only if all M_i are of type 2 (Shastri and Zvengrowski, 1979). It is easy to compute $[M_1 \# M_2, P^3]$ given that $[M_1, P^3] \approx Z \oplus Z_2^p$ and $[M_2, P^3] \approx Z \oplus Z_2^q$, with p and q known. There are three cases:
(a) M_1, M_2 both type $2 \Rightarrow [M_1 \# M_2, P^3] \approx Z \oplus Z_2^{p+q}$
(b) mixed types $\Rightarrow [M_1 \# M_2, P^3] \approx Z \oplus Z_2^{p+q}$
(c) M_1, M_2 both type $1 \Rightarrow [M_1 \# M_2, P^3] \approx Z \oplus Z_2^{p+q+1}$
Note that an infinite family of irreducible 3-manifolds, all of type 1 and having the same homology as P^3, is furnished by the thesis of López de Medrano (1971, p. 25). We remark that the classification scheme for $S(p)$, namely, $Z \oplus Z_2^p$, has already been anticipated by Finkelstein (1978), and some work has been done on the one-handle case (Finkelstein and Misner, 1962).

It was previously pointed out that the metric

$$g_{\mu\nu} = \delta_{\mu\nu} - 2\phi_\mu \phi_\nu$$

provides an example of a 1-kink metric for $M = S^3$. Constructing such examples for more complicated M becomes rather involved and we shall be content with examining the $M = S^1 \times S^2$ case. Here the homotopy classes of metrics are labeled by a pair $(n; t) \in Z \oplus Z_2$. Let us denote points of S^2 by $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, where $\sum \mu_i^2 = 1$, and let β be an angular variable, $0 \leqslant \beta \leqslant 2\pi$ with 0 and 2π identified. The pair $(\beta, \boldsymbol{\mu})$ then represents a point in $S^1 \times S^2$. As base point we take $\beta = 0, \boldsymbol{\mu} = (0,0,1)$. Points $(\phi_1, \phi_2, \phi_3, \phi_4) = (\boldsymbol{\phi}, \phi_4) \in S^3$ can be conveniently represented by a 2×2 unitary matrix $U = I\phi_4 + i\boldsymbol{\tau} \cdot \boldsymbol{\phi}$, where I is the unit matrix and $\{\tau_i\}$ are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$[U]$ is then a point of P^3, and Table II lists various unitary matrices representing maps $S^1 \times S^2 \to P^3$. By identifying $P^3 = M_{4,1}$ as the subspace $S_{4,1} \cap O(4)$ of $S_{4,1}$ (Steenrod, 1951, §40.2) and choosing an explicit framing for $S^1 \times S^2$, any of the maps in Table II can be reinterpreted as a Lorentz metric for the case $M = S^1 \times S^2$.

TABLE II. Maps for the Metric for $M=S^1\times S^2$

$(n;t)$	$[U]=[I\phi_4+i\tau\cdot\phi]$
(0,0)	$[U_{0,0}]=[I]$
(0,1)	$[U_{0,1}]=[I\cos\frac{1}{2}\beta+i\tau_3\sin\frac{1}{2}\beta]$
(0,2)	$[U_{0,2}]=[U_{0,1}U_{0,1}]=[I\cos\beta+i\tau_3\sin\beta]$
(1,1)	$[U_{1,1}]=[I\cos\frac{1}{2}\beta+i\tau\cdot\mu\sin\frac{1}{2}\beta]$
(1,0)	$[U_{1,0}]=[U_{1,1}U_{0,1}]$
(2,2)	$[U_{2,2}]=[U_{1,1}U_{1,1}]$

7. PHYSICAL INTERPRETATION AND CONCLUSIONS

For a parallelizable space-time manifold $\mathfrak{M}^4$ that is a bundle space with no closed time lines and with any closed, connected, orientable 3-manifold M as base, the procedure for determining all possible homotopy classes of metrics has been specified. This is equivalent to calculating $[M,P^3]$ for all M, which was shown to have a particularly simple form, namely, $[M,P^3]\approx Z\oplus Z_2^l$. The value of l, the number of copies of Z_2, depends partly on whether degree-1 maps $M\to P^3$ are possible (in which case M is called type 1) or not (in which case M is called type 2, and all maps are of even degree). The homotopy class of a metric can be labeled by $(n;t_1,\ldots,t_l)\in Z\oplus Z_2^l$, where $t_i=0$ or 1 (or ± 1), $i=1,2,\ldots,l$. The kink number $n\in Z$ is clearly the important label and counts the number of "particles" or "structures" present. It is possible that the latter will find applications in astrophysics and black hole theory (Finkelstein and McCollum, 1975; Finkelstein, 1978); yet the fermionlike properties of the kinks suggest possible applications in elementary particle physics, too. It was shown that half-odd-integer spin occurs if n is odd. The fact that there is a unique counting number $n\in Z$ and fermion behavior for *all possible choices* of M is very striking, and it would seem odd if Nature should not make use of this somehow.

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REFERENCES

Barcus, W. (1954). *Quarterly Journal of Mathematics (Oxford)*, **5**, 150.

Cartan, H., and Eilenberg, S. (1956). *Homological Algebra*. Princeton University Press, Princeton, New Jersey.

Clarke, C. J. S. (1979). "Approximately Straight Kink Theories." University of York Preprint. Submitted to the *International Journal of Theoretical Physics*.

Dowker, J. S. (1972). *Journal of Physics A*, **5**, 936.

Finkelstein, D. (1966). *Journal of Mathematical Physics*, **7**, 1218.

Finkelstein, D. (1978). The Delinearization of Physics, in *Proceedings of the Symposium on the Foundations of Modern Physics: Loma-Koli, Finland, August 11–18, 1977*, V. Karimäki, ed. Publications of the University of Joensuu, Finland.

Finkelstein, D., and McCollum, G. (1975). *Journal of Mathematical Physics*, **16**, 2250.

Finkelstein, D., and Misner, C. W. (1959). *Annals of Physics*, **6**, 230.

Finkelstein, D., and Misner, C. W. (1962). "Further Results in Topological Relativity," in *Les Theories Relativistes de la Gravitation: Royaumont, June 21–27, 1959*, A. Lichnerowicz and M. A. Tonnelat, eds. Centre National de la Recherche Scientifique, Paris.

Finkelstein, D., and Rubinstein, J. (1968). *Journal of Mathematical Physics*, **9**, 1762.

Fisher, A. E. (1970). "The Theory of Superspace," in *Relativity—Proceedings of the Relativity Conference in the Midwest: Cincinnati, Ohio, June 2–6, 1969*, M. Carmeli, S. I. Fickler, and L. Witten, eds. Plenum Press, New York.

Hu, S. -T. (1959). *Homotopy Theory*. Academic Press, New York.

Lichnerowicz, A. (1968). "Topics on Space-Times," in *Battelle Rencontres in Mathematics and Physics: Seattle, 1967*, C. M. DeWitt and J. A. Wheeler, eds. Benjamin, New York.

López de Medrano, S. (1971). *Involutions on Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 59. Springer-Verlag, Berlin.

Milgram, R. J., and Zvengrowski, P. (1974). *Proceedings of the London Mathematical Society*, **28**, 671.

Mosher, R. E., and Tangora, M. C. (1968). *Cohomology Operations and applications in Homotopy Theory*. Harper and Row, New York.

Pak, N. K., and Tze, H. C. (1979). *Annals of Physics*, **117**, 164.

Shastri, A. R., and Zvengrowski, P. (1979). "3-Manifolds Admitting Degree 1 Maps onto RP^3." University of Calgary, Yellow Series Preprint.

Siebenmann, L. C. (1969). *Transactions of the American Mathematical Society*, **142**, 201.

Skyrme, T. H. R. (1971). *Journal of Mathematical Physics*, **12**, 1735.

Spanier, E. H. (1966). *Algebraic Topology*. McGraw-Hill, New York.

Steenrod, N. (1951). *Topology of Fibre Bundles*. Princeton University Press, Princeton, New Jersey.

Williams, J. G. (1970). *Journal of Mathematical Physics*, **11**, 2611.

Williams, J. G., and Zia, R. K. P. (1973). *Journal of Physics A*, **6**, 1.

Williams, J. G., and Zvengrowski, P. (1977). *International Journal of Theoretical Physics*, **16**, 755.